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The Kramer–Neugebauer limit and Nakamura's conjecture

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Abstract
The Kramer–Neugebauer limit of their solution of the Ernst equation to give the Tomimatsu–Sato class of solutions is obtained. This gives insight into Nakamura's conjecture.

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1. Introduction

Kramer and Neugebauer [1] have indicated how it may be possible to obtain the Tomimatsu–Sato [2] family of solutions from their solution of the Ernst equation by a limiting process. However, the form of their solution makes the carrying out of this procedure in the general case an awesome prospect. By using an equivalent form [3] of their solution and by replacing certain sums by contour integrals, we show that the limit is easily effected. When this is achieved, we arrive at a form of solution similar to that conjectured by Nakamura [4, 5].

2. The Kramer–Neugebauer limit

Let

$$E_n = \left(\lambda_{rs}^{(n)}\right)_{n\times n} \tag{2.1}$$

$$E_{n-1} = \left(\lambda_{rs}^{(n)}\right)_{(n-1)\times(n-1)} \tag{2.2}$$

where

$$\lambda_{rs}^{(n)} = \sum_{\kappa=1}^{2n} \frac{(z+c_\kappa)^{r+s-2}\,\mathrm{e}^{\mathrm{i}\theta_\kappa} F(c_\kappa)}{\tau_\kappa} \tag{2.3}$$

$$F(c_\kappa) = \sqrt{\rho^2 + (z+c_\kappa)^2} \tag{2.4}$$

$$\tau_\kappa = \prod_{\substack{t=1\\ t\neq\kappa}}^{2n} (c_\kappa - c_t) \tag{2.5}$$

then the Ernst's potential function ξ_n is given by [3] (p 301, 6.10.61)

$$\xi_n = \left(\frac{2}{\rho}\right)^{2n-1} V_{2n}(c)\frac{P_{n-1}}{P_n} \tag{2.6}$$

$$= \frac{\det E_{n-1}}{\det E_n}\text{([3], p 297, 6.10.33; p 300, 6.10.56)} \tag{2.7}$$

choosing

$$\theta_{2\kappa-1} = \theta \qquad \theta_{2\kappa} = \pi - \theta \qquad 1 \leqslant \kappa \leqslant n$$

$$\lambda_{rs}^{(n)} = \upsilon_{rs}^{(1)}\,\mathrm{e}^{\mathrm{i}\theta} - \upsilon_{rs}^{(2)}\,\mathrm{e}^{-\mathrm{i}\theta} \tag{2.8}$$

$$\upsilon_{rs}^{(1)} = \sum_{\kappa=1}^{n} \frac{(z + c_{2\kappa-1})^{r+s-2} F(c_{2\kappa-1})}{\tau_{2\kappa-1}} \tag{2.9}$$

$$\upsilon_{rs}^{(2)} = \sum_{\kappa=1}^{n} \frac{(z + c_{2\kappa})^{r+s-2} F(c_{2\kappa})}{\tau_{2\kappa}}. \tag{2.10}$$

It is now required to put $c_{2\kappa-1} = 1$, $c_{2\kappa} = -1$ for $1 \leqslant \kappa \leqslant n$. However, since this would make the τ zero (except when $n = 1$), it would be necessary to take limits. This can be avoided by noting that the sums on the right-hand sides of (2.9) and (2.10) can be written as contour integrals, using Cauchy's theorem,

$$\upsilon_{rs}^{(\alpha)} = \frac{1}{2\pi\mathrm{i}}\int_{C_\alpha} \frac{(z+\omega)^{r+s-2}F(\omega)}{\prod_{\kappa=1}^{2n}(\omega - c_\kappa)}\,\mathrm{d}\omega \tag{2.11}$$

$$F(\omega) = \sqrt{\rho^2 + (\omega + z)^2} \qquad \alpha = 1, 2. \tag{2.12}$$

Where the contour C_1 contains the poles $\omega = c_{2k-1}$ but not the poles $\omega = c_{2\kappa}$, and C_2 contains poles $\omega = c_{2\kappa}$ but not the poles $\omega = c_{2\kappa-1}$ for $\kappa = 1, 2, 3, \ldots, n$.
Both contours exclude the branch points $\omega = -z \pm \mathrm{i}\rho$ of $F(\omega)$.
We can now put $c_{2\kappa-1} = 1$, $c_{2\kappa} = -1$, $1 \leqslant \kappa \leqslant n$ to give

$$\upsilon_{rs}^{(\alpha)} = \frac{1}{2\pi\mathrm{i}}\int_{C_\alpha} \frac{(z+\omega)^{r+s-2}F(\omega)}{(\omega+1)^n(\omega-1)^n}\,\mathrm{d}\omega \tag{2.13}$$

C_1 now contains the pole at $\omega = 1$, but not the pole at $\omega = -1$, and C_2 contains $\omega = -1$ but not the pole at $\omega = 1$. Both C_1 and C_2 exclude the branch points of $F(\omega)$. It is shown in appendix A.1 that we can rewrite ξ_n in the form

$$\xi_n = \frac{D_{11}^{(n)}}{D_n} \tag{2.14}$$

$$D_n = |d_{rs}|_n \tag{2.15}$$

$$D_{11}^{(n)} = \text{cofactor of } d_{11} \text{ in } D_{11} \tag{2.16}$$

$$d_{rs} = p\gamma_{rs}^{-} + \mathrm{i}q\gamma_{rs}^{+} \tag{2.17}$$

$$\gamma_{rs}^{-} = \gamma_{rs}^{(1)} - \gamma_{rs}^{(2)} \tag{2.18}$$

$$\gamma_{rs}^{+} = \gamma_{rs}^{(1)} + \gamma_{rs}^{(2)} \tag{2.19}$$

$$\gamma_{rs}^{(\alpha)} = \frac{1}{2\pi\mathrm{i}}\int_{C_\alpha} \frac{F(\omega)}{(\omega-1)^r(\omega+1)^s}\,\mathrm{d}\omega \tag{2.20}$$

$$\alpha = 1, 2 \tag{2.21}$$

$$p^2 + q^2 = 1. \tag{2.22}$$

3. Nakamura's conjecture

If we change from coordinates ρ, z to x, y defined by

$$\rho = \sqrt{(x^2 - 1)(1 - y^2)} \tag{3.1}$$

$$z = xy \tag{3.2}$$

then Nakamura's conjecture is

$$\xi_n = \frac{U_{11}^{(n)}}{U_n} \tag{3.3}$$

$$U_n = |u_{rs}|_n \tag{3.4}$$

$$U_n^{(11)} = \text{cofactor of } u_{11} \text{ in } U_n \tag{3.5}$$

$$u_{rs} = L_+^{r-1} L_-^{s-1} \phi \tag{3.6}$$

$$L_\pm = (1 - x^2)\frac{\partial}{\partial x} \pm (1 - y^2)\frac{\partial}{\partial y} \tag{3.7}$$

$$\phi = px + \mathrm{i}qy \tag{3.8}$$

$$p^2 + q^2 = 1. \tag{3.9}$$

Let

$$f_m = \theta^m x \qquad m \geqslant 1 \qquad \theta = (1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x} \tag{3.10}$$

$$= x \qquad m = 0. \tag{3.11}$$

Then

$$L_-\phi = pf_1(x) - \mathrm{i}qf_1(y) \qquad L_-^2\phi = pf_2(x) + \mathrm{i}qf_2(y).$$

Generally then $L_-^{s-1} = pf_{s-1}(x) + \mathrm{i}q(-1)^{s-1} f_{s-1}(y)$

$$\therefore u_{rs} = L_+^{r-1} L_-^{s-1}\phi = pf_{r+s-2}(x) + \mathrm{i}q(-1)^{s-1} f_{r+s-2}(y). \tag{3.12}$$

It is shown in appendix A.2 for $n = 1, 2, 3$ that our ξ_n agrees with (3.3) and (3.12), thus proving Nakamura's conjecture for $n = 1, 2, 3$. It is also shown in appendix A.3 that if Nakamura's conjecture is correct then it satisfies the necessary and sufficient conditions for it to be the Tomimatsu–Sato class of solutions.

In order to prove the conjecture generally it would be necessary to generalize the matrices A and B of appendix A.2.

Acknowledgment

I wish to thank Barry Martin [7] who has checked all of the formulae in appendix A.2 using MAPLE.

Appendix

A.1.

Theorem 1. *Let $X = (x_{rs})$, $Y = (y_{rs})$ and $Z = (z_{rs})$ be invertible square matrices of order n. Then if $W = XYZ$*

$$y^{rs} = \sum_{\kappa=1}^{n}\sum_{m=1}^{n} x_{\kappa r}\omega^{\kappa m} z_{sm}$$

where

$$y^{rs} = \frac{Y_{rs}}{|Y|} \qquad \omega^{rs} = \frac{W_{rs}}{|W|}$$

$$Y_{rs} = \text{cofactor of } y_{rs} \text{ in } Y$$

$$W_{rs} = \text{cofactor of } \omega_{rs} \text{in } W.$$

In particular if

$$(a) \quad x_{rr} = 1 \qquad x_{rs} = 0 \qquad r < s$$

$$z_{rr} = 1 \qquad z_{rs} = 0 \qquad s < r$$

then

$$Y_{nn} = W_{nn}$$

$$(b) \quad x_{r1} = \delta_{1r} \qquad z_{1s} = \delta_{1s}$$

$$(\delta_{rs} = 1 \quad r = s \quad \delta_{rs} = 0 \quad r \neq s)$$

then

$$y^{11} = \omega^{11}.$$

Proof. Since

$$W = XYZ$$

taking the inverse

$$W^{-1} = Z^{-1}Y^{-1}X^{-1}$$

$$\therefore Y^{-1} = ZW^{-1}X$$

$$\therefore (Y^{-1})^T = X^T(W^{-1})^T Z^T$$

$$\therefore y^{rs} = \sum_{\kappa=1}^{n}\sum_{m=1}^{n} x_{\kappa r}\omega^{\kappa m} z_{sm}$$

$$(a) \quad y^{nn} = \sum_{\kappa=1}^{n}\sum_{m=1}^{n} x_{\kappa n}\omega^{\kappa m} z_{nm} = x_{nn}\omega^{nn} z_{nn} = \omega^{nn}.$$

Also

$$|X| = |Y| = 1 \qquad \therefore Y_{nn} = W_{nn}$$

$$(b) \quad y^{11} = \sum_{\kappa=1}^{n}\sum_{m=1}^{n} x_{\kappa 1}\omega^{\kappa m} z_{1m} = x_{11}\omega^{11} z_{11} = \omega^{11}.$$

Let

$$R_n(a) = (\alpha_{rs}(a))_{n\times n}$$

where

$$\alpha_{rs}(a) = \binom{r-1}{s-1} a^{r-s} \qquad 1 \leqslant s \leqslant r$$
$$= 0 \qquad r < s.$$

Then

$$\det E_n = \det(R_n(a) E_n (R_n(b))^T)$$
$$= \det(\beta_{rs})_{n\times n}$$

where

$$\beta_{rs} = \sum_{\kappa=1}^{r}\sum_{m=1}^{s} \binom{r-1}{\kappa-1}\binom{s-1}{m-1} a^{r-\kappa} b^{s-m} \left(\upsilon_{\kappa m}^{(1)}\,\mathrm{e}^{\mathrm{i}\theta} - \upsilon_{\kappa m}^{(2)}\,\mathrm{e}^{-\mathrm{i}\theta}\right).$$

But

$$\sum_{\kappa=1}^{r}\sum_{m=1}^{s} \binom{r-1}{\kappa-1}\binom{s-1}{m-1} a^{r-\kappa} b^{s-m} \upsilon_{\kappa m}^{(\alpha)}$$
$$= \frac{1}{2\pi\mathrm{i}} \int_{C_\alpha} \frac{(\omega+z+a)^{r-1}(\omega+z+b)^{s-1}F(\omega)}{(\omega-1)^n(\omega+1)^n}\,\mathrm{d}\omega$$
$$= \frac{1}{2\pi\mathrm{i}} \int_{C_\alpha} \frac{F(\omega)}{(\omega-1)^{n-r+1}(\omega+1)^{n-s+1}}\,\mathrm{d}\omega$$

choosing

$$a = -1 - z \qquad b = 1 - z$$
$$\therefore \beta_{rs} = d_{(n-r+1)(n-s+1)}$$

where

$$d_{rs} = \gamma_{rs}^{(1)}\,\mathrm{e}^{\mathrm{i}\theta} - \gamma_{rs}^{(2)}\,\mathrm{e}^{-\mathrm{i}\theta}$$
$$\gamma_{rs}^{(\alpha)} = \frac{1}{2\pi\mathrm{i}} \int_{C_\alpha} \frac{F(\omega)}{(\omega-1)^r(\omega+1)^s}\,\mathrm{d}\omega$$
$$\det E_n = |d_{(n-r+1)(n-s+1)}|_n$$
$$= |\delta_{(n-r+1)s}|_n |d_{(n-r+1)(n-s+1)}|_n |\delta_{(n-s+1)r}|_n$$
$$= |d_{rs}|_n$$
$$= D_n.$$

By theorem 1(a)

$$\det(E_{n-1}) = |d_{(n-r+1)(n-s+1)}|_{n-1}$$
$$= |\delta_{(n-r)s}|_{n-1} |d_{(n-r+1)(n-s+1)}|_{n-1} |\delta_{(n-s)r}|_{n-1}$$
$$= |d_{(r+1)(s+1)}|_{n-1}$$
$$= D_{11}^{(n)}.$$

Now

$$d_{rs} = \gamma_{rs}^{(1)}\mathrm{e}^{\mathrm{i}\theta} - \gamma_{rs}^{(2)}\,\mathrm{e}^{-\mathrm{i}\theta}$$
$$= \gamma_{rs}^{(1)}(\cos\theta + \mathrm{i}\sin\theta) - \gamma_{rs}^{(2)}(\cos\theta - \mathrm{i}\sin\theta)$$
$$= \gamma^{-}{}_{rs}\cos\theta + \mathrm{i}\sin\theta\,\gamma_{rs}^{+}$$
$$\gamma^{-}{}_{rs} = \gamma_{rs}^{(1)} - \gamma_{rs}^{(2)}$$
$$\gamma_{rs}^{+} = \gamma_{rs}^{(1)} + \gamma_{rs}^{(2)}.$$

Putting $p = \cos\theta, q = \sin\theta$ so that $p^2 + q^2 = 1$

$$d_{rs} = p\gamma_{rs}^{-} + \mathrm{i}q\gamma_{rs}^{+}. \qquad \square$$

A.2.

Using equations (2.14)–(2.22) inclusive and (3.12) gives

$$d_{11} = u_{11} \qquad d_{12} = \frac{u_{12}}{2(y-x)}$$

$$d_{21} = \frac{u_{21}}{2(x+y)} \qquad d_{22} = \frac{u_{22}}{4(y^2-x^2)}.$$

In order to obtain these results it is necessary to define

$$F(1) = x + y \qquad \text{and} \qquad F(-1) = x - y.$$

$$\therefore \xi_1 = \frac{U_{11}^{(1)}}{U_1} \qquad \left(U_{11}^{(1)} = 1\right)$$

$$\xi_2 = \frac{U_{11}^{(2)}}{U_2}.$$

In order to obtain the results for $n = 3$ it is necessary to transform the determinant D_3.

Let $D = (d_{rs})_{3\times3}$ and form the matrix product $ADB = (d_{rs}^{*})_{3\times3,}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2(xy+1)}{(x+y)} & 4(x+y) \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2(-xy+1)}{(x-y)} \\ 0 & 0 & 4(y-x) \end{bmatrix}.$$

This gives

$$d_{11}^{*} = u_{11} \qquad d_{12}^{*} = \frac{u_{12}}{2(y-x)} \qquad d_{13}^{*} = \frac{u_{13}}{2(y-x)}$$

$$d_{21}^{*} = \frac{u_{21}}{2(x+y)} \qquad d_{22}^{*} = \frac{u_{22}}{4(y^2-x^2)} \qquad d_{23}^{*} = \frac{u_{23}}{4(y^2-x^2)}$$

$$d_{31}^{*} = \frac{u_{31}}{2(x+y)} \qquad d_{32}^{*} = \frac{u_{32}}{4(y^2-x^2)} \qquad d_{33}^{*} = \frac{u_{33}}{4(y^2-x^2)}$$

$\therefore$ by theorem 1(b) we have

$$\xi_3 = \frac{D_{11}^{(3)}}{D_3} = \frac{D_{11}^{(3)*}}{D_3^{*}} = \frac{U_{11}^{(3)}}{U_3}.$$

A.3.

In the notation of [6] (p 464, equation (8)) the necessary and sufficient condition for a solution to be a member of the Tomimatsu–Sato class is

$$F_x F_y + J_x J_y = 0 \qquad \text{(A.3.1)}$$

where suffix x denotes the partial differentiation with respect to x, and suffix y denotes the partial differentiation with respect to y. Where $\xi = J + \mathrm{i}F$.

In terms of ξ(A.3.1) reads

$$\xi_x\bar{\xi}_y + \bar{\xi}_x\xi_y = 0 \qquad \text{(A.3.2)}$$

where the bar denotes complex conjugate.

(A.3.2) can be rewritten as

$$L_+\xi L_+\bar{\xi} - L_-\xi L_-\bar{\xi} = 0. \tag{A.3.3}$$

Theorem 2. *Let*

$$\xi = \frac{U_{11}^{(n)}}{U_n} \tag{A.3.4}$$

$$U_n = |u_{rs}|_n \tag{A.3.5}$$

$$U_{11}^{(1)} = \text{cofactor of } u_{11} \text{ in } U_n \tag{A.3.6}$$

$$u_{rs} = pf_{r+s-2}(x) + \mathrm{i}q(-1)^{s-1} f_{r+s-2}(y) \tag{A.3.7}$$

then

$$U_n^2 L_+\xi = \bar{U}_n^2 L_-\bar{\xi}. \tag{A.3.8}$$

Proof. From (A.3.7)

$$u_{r(s+2)} = u_{(r+2)s} \tag{A.3.9}$$

$$\bar{u}_{rs} = u_{(r+1)(s-1)} \qquad s \geqslant 2 \tag{A.3.10}$$

$$\bar{u}_{rs} = u_{(r-1)(s+1)} \qquad r \geqslant 2. \tag{A.3.11}$$

From (A.3.10) or (A.3.11)

$$\bar{U}_{1n}^{(n)} = U_{n1}^{(n)} \tag{A.3.12}$$

$$\therefore \bar{U}_{(n+1)1}^{(n+1)} = U_{1(n+1)}^{(n+1)}. \tag{A.3.13}$$

Now

$$\begin{aligned}
U_n^2 L_+\xi &= -\sum_{r=1}^{n}\sum_{s=1}^{n} L_+(u_{rs}) U_{1s}^{(n)} U_{r1}^{(n)} \\
&= -\sum_{r=1}^{n} U_{r1}^{(n)} \sum_{s=1}^{n} u_{(r+1)s} U_{1s}^{(n)} \\
&= -U_{n1}^{(n)} \sum_{s=1}^{n} u_{(n+1)s} U_{1s}^{(n)} \\
&= U_{n1}^{(n)} U_{1(n+1)}^{(n+1)} \\
U_n^2 L_-\xi &= -\sum_{r=1}^{n}\sum_{s=1}^{n} L_-(u_{rs}) U_{1s}^{(n)} U_{r1}^{(n)} \\
&= -\sum_{s=1}^{n} U_{1s}^{(n)} \sum_{r=1}^{n} u_{r(s+1)} U_{r1}^{(n)} \\
&= -U_{1n}^{(n)} \sum_{s=1}^{n} u_{r(n+1)} U_{r1}^{(n)} \\
&= U_{1n}^{(n)} U_{(n+1)1}^{(n+1)}. \\
\therefore \bar{U}_n^2 L_-\bar{\xi} &= \bar{U}_{1n}^{(n)} \bar{U}_{(n+1)1}^{(n+1)} \\
&= U_{n1}^{(n)} U_{1(n+1)}^{(n+1)} \\
&= U_n^2 L_+\xi.
\end{aligned}$$

From theorem 2

$$\frac{L_-\bar{\xi}}{L_+\xi} = \frac{U_n^2}{\bar{U}_n^2}.$$

Taking complex conjugates gives

$$\frac{L_-\xi}{L_+\bar{\xi}} = \frac{\bar{U}_n^2}{U_n^2}$$

$$= \frac{L_+\xi}{L_-\bar{\xi}}.$$

$$\therefore L_+\xi L_+\bar{\xi} - L_-\xi L_-\bar{\xi} = 0$$

which is condition (A.3.3). Thus if Nakamura's conjecture is correct then it is a member of the Tomimatsu–Sato class. □

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